Fermions, scalars and Randall-Sundrum gravity on domain-wall branes

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(Dated: February 1, 2008)

Abstract

We analyse the general features of localisation of fermions and scalars in smoothed field-theoretical versions of the type 2 Randall- Sundrum braneworld model. A scalar field domain-wall forms the brane, inducing warped gravity, and we study the mass spectra of the matter fields in the dimensionally reduced theory. We demonstrate explicitly that both scalar and fermion fields exhibit a continuum of properly normalisable modes starting at zero mass. If discrete bound modes are present in the gravity-free case, these become resonances in the continuum, while off-resonant modes are highly suppressed on the brane. We describe briefly how another scalar field can be used to break a symmetry on the domain-wall while leaving it unbroken far from the wall, as has already been done in the flat space case. Finally we present numerical calculations for a toy model which demonstrates the decoupling of continuum modes at low energies, so the theory becomes four dimensional.

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arXiv:0705.1391v4 [hep-ph] 25 Sep 2007

I. INTRODUCTION

Over the last decade there has been a lot of research into the possibility that our observable $3 + 1$-dimensional universe resides on a defect in a higher-dimensional bulk spacetime. If this is the case, we should be able to construct a model which explains why physics at low energies is insensitive to the extra dimensions. It has been known for some time that bulk fermions coupled to a scalar field domain-wall have a spectrum containing a zero mode which is localised to the wall [1]. In the case of a $4 + 1$-dimensional bulk, this zero mode fermion is chiral and separated by a mass gap from the massive modes, making it well suited to model building efforts. Scalar fields can be localised on the wall in a similar fashion, and in particular, the canonical Mexican hat potential can be generated for the lowest lying mode [2].

The gravitational part of the puzzle found a solution in the type 2 Randall-Sundrum model (RS2) [3], where the metric is warped to ensure $3+1$-dimensional gravity is reproduced on an infinitely thin defect. Extensions to smoothed out defects induced by scalar fields have since been constructed (see eg. [4, 5, 6, 7, 8, 9]). When combining this warped gravity set up with the fermion and scalar localisation, it is important to ensure that gravity doesn't destroy the desirable features of the matter localisation.

For the case of scalar fields, it was first noted in [10] that with gravity included, the scalar continuum modes can begin at zero energy. In [11] it was shown that this can be true for fermions also, and that introducing a $4 + 1$-dimensional mass term can produce massive, localised, meta-stable states. Due to their coupling to the low lying continuum, these states can tunnel into the bulk and have a finite lifetime. Ref. [12] considered a specific model and determined the full mass spectrum of the meta-stable, or quasi-localised, modes and demonstrated that their lifetime could be made longer the age of the universe.

In this paper we present a general analytic argument which demonstrates explicitly that in the presence of RS2-like gravity, the continuum matter modes always begin at zero energy. It is also demonstrated that the discrete modes present in the gravity-free theory show up in the warped case as resonances of the domain-wall trapping potential, which represent the quasi-localised states. It can be argued that in the presence of interactions, the low-energy non-resonant continuum modes will couple only very weakly to modes localised on the brane. Thus $3+1$-dimensional physics can still be reproduced at low energies. These considerations

are important for realistic model-building [13].

The paper is structured as follows. In Section II an overview of localised fermions in the gravity-free case is given, followed by an analysis with gravity included in Section III. This later section demonstrates that the fermion mass spectrum has a continuum beginning at zero, and that these massive modes can be properly normalised in the presence of the warped metric. In Section IV we repeat the gravitational analysis for a coupled scalar field and show that while the continuum modes begin at zero mass, it is still possible to obtain discrete modes with tachyonic mass, allowing one to realise the Higgs mechanism. In Section V we consider a specific toy model to demonstrate more clearly the effect of gravity, and provide numerical support to the claim that the gravity induced continuum modes are only weakly coupled to a brane-localised zero mode. Section VI concludes.

II. FERMIONS IN THE GRAVITY-FREE CASE

We begin with a brief review of the situation without gravity. Consider a 5D model containing a number of scalar fields Φ_j. These will form our classical background. The action is,

$$\mathcal{S}_{bg} = \int d^4x \int dy \left\{ \frac{1}{2} \partial^A \Phi_j \partial_A \Phi_j - V(\Phi) \right\}. \tag{1}$$

Here $A = (0, 1, 2, 3, 5)$ is a 5D Lorentz index and repeated indices are summed. The only additional assumptions we make are that V has a $\mathbb{Z}_2$ symmetry $\Phi_j \to -\Phi_j \ \forall \ j$, such that the global minimum of V is at least doubly degenerate, and attained for (say) $\Phi_j = \pm\Phi_j^{\min}$. We also require this $\mathbb{Z}_2$ to be independent of any continuous symmetries of the theory.

We suppose we have a solution $\phi_j(y)$ depending only on the coordinate y, satisfying the boundary conditions $\phi_j \to \pm\Phi_j^{\min}$ as $y \to \pm\infty$. Such a solution is topologically stable, and can be used as a classical background for a quantum field theory [14].

Now introduce a fermion field Ψ into the model, and Yukawa-couple it to the domain-wall. Its action will be,

$$\mathcal{S}_\Psi = \int d^4x \int dy \left\{ i\overline{\Psi}\Gamma^A\partial_A\Psi - g_j\Phi_j\overline{\Psi}\Psi \right\}, \tag{2}$$

where the g_j are Yukawa coupling constants and $\Gamma^\mu = \gamma^\mu, \Gamma^5 = -i\gamma^5$ with $\gamma^{\mu,5}$ the usual 4D Dirac matrices and chirality operator, respectively. The action of the $\mathbb{Z}_2$ symmetry is extended to include $y \to -y$ and $\Psi \to \Gamma^5\Psi$. For simplicity we have also imposed a global

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$U(1)$ symmetry $\Psi \to e^{i\theta}\Psi$, to forbid a term $g_j'\Phi\overline{\Psi}\Psi^c + h.c.$. Therefore, in the classical background discussed above, the Dirac equation will be,

$$\left[i\Gamma^A\partial_A - g_j\phi_j(y)\right]\Psi(x^\mu, y) = 0. \tag{3}$$

To solve this equation, we separate variables by expanding Ψ in a generalised Fourier series,

$$\Psi(x^\mu, y) = \sum_n \left[f_L^n(y)\psi_L^n(x^\mu) + f_R^n(y)\psi_R^n(x^\mu)\right], \tag{4}$$

where the $\psi_{L,R}$ are left- and right-handed 4D spinors, and are treated independently due to the association of Γ^5 with y. The sum over n generally includes an integral over continuum parts. We choose our basis functions such that the $\psi_{L,R}$ satisfy the 4D Dirac equation,

$$i\gamma^\mu\partial_\mu\psi_L^n = m_n\psi_R^n \quad \text{and} \quad i\gamma^\mu\partial_\mu\psi_R^n = m_n\psi_L^n.$$

We can then solve Eq. (3) for $f_{L,R}^n$. In the $m_0 = 0$ case, we get the decoupled equations (hereafter, primes denote differentiation with respect to y),

$$\begin{aligned}
f_L^{0\prime} + g_j\phi_j f_L^0 &= 0, \\
f_R^{0\prime} - g_j\phi_j f_R^0 &= 0.
\end{aligned} \tag{5}$$

The solutions are,

$$f_{L,R}^0(y) \propto \exp\left(\mp\int^y d\tilde{y}\, g_j\phi_j(\tilde{y})\right). \tag{6}$$

The boundary conditions satisfied by the ϕ_j then give generically that if $g_j\Phi_j^{\min} > 0$ the left-handed component of Ψ is localised near $y = y_0$, where $g_j\phi_j(y_0) = 0$ (the existence of such a point follows from the boundary conditions), while the right-handed component is non-normalisable, and thus unphysical. This is the well-known result that a domain-wall localises chiral fermions [1]. More interesting is the case $m_n > 0$. We then get the following equation for $f_{L,R}$,

$$-f_{L,R}^{n\,\prime\prime} + W_\mp f_{L,R}^n = m_n^2 f_{L,R}^n, \tag{7}$$

where $W_\pm = (g_j\phi_j)^2 \pm g_j\phi_j'$. Eq. (7) is just a Schrödinger equation with eigenvalue m_n^2. The potential W is a finite well, and $W \to (g_j\Phi_j^{\min})^2 > 0$ as $y \to \pm\infty$. Therefore the normalisable solutions will consist of a number of discrete bound states, as well as a continuum starting at some non-zero energy[1]. Thus the 4D fermion spectrum contains a massless left-handed

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[1] The continuum modes will only be delta-function normalisable.

particle, a finite number of massive Dirac particles with discrete masses, and a continuum of massive Dirac particles beginning at $m_{\text{cont}} = g_j \Phi_j^{\text{min}}$. Explicit solutions to one specific model can be found in [2].

III. INCLUDING GRAVITY

To include gravity in these models, we simply add the Einstein-Hilbert term to the action, and minimally couple other fields to gravity as usual, to obtain the background action

$$\mathcal{S}_{bg} = \int d^4x \int dy \, \sqrt{G} \Big\{ -2M^3 R - \Lambda + \frac{1}{2} G^{MN} \partial_M \Phi_j \partial_N \Phi_j - V(\Phi) \Big\}, \qquad (8)$$

where G_{MN} is the 5D metric, G its determinant, M the 5D Planck mass, R the 5D Ricci scalar, and Λ the bulk cosmological constant.

Now we must solve the coupled Einstein and Klein-Gordon equations associated with (8). We again assume that the solution $\phi_j(y)$ depends only on y, and satisfies the same boundary conditions as earlier. The resulting equations have been solved by various authors [4, 5, 6, 7, 8, 9] and here we merely point out the general features. The solution for the metric is given by,

$$ds^2 = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2 \qquad (9)$$

where σ is a smooth even function of y, and $\sigma \to \mu|y|$ as $|y| \to \infty$, where μ is some mass scale. In the RS2 model, the domain-wall is instead a delta-function brane, and $\sigma(y) = \mu|y|$ everywhere. The effective 4D graviton spectrum, corresponding to fluctuations around the metric in Eq. (9), was shown to consist of a single massless mode, followed by a continuum of massive modes starting arbitrarily close to zero mass [3]. Contrary to naïve expectations, this does not contradict the assertion that the low-energy theory is 4 dimensional. In fact, the integrated effect of the continuum modes at the position of the brane is negligible at low energies, due to the suppression of their wavefunctions near the brane. Csáki et al. have shown that the same result holds in the present context of smooth RS-like spacetimes [15]. We will demonstrate in this and the next section that similar statements are true for fermions and scalar fields coupled to the above background.

With gravity included, the formulation of the Dirac Lagrangian requires the introduction of the vielbein V_A^N (here A is an 'internal' Lorentz index) and the spin connection ω_N, given

for the metric in Eq. (9) by,

$$V_A^\mu = \delta_A^\mu e^\sigma \qquad \omega_\mu = \frac{i}{2}\sigma' e^{-\sigma}\gamma_\mu\gamma^5$$
$$V_A^5 = \delta_A^5 \qquad \omega_5 = 0. \tag{10}$$

These yield the spin-covariant derivative $D_N = \partial_N + \omega_N$, and curved space gamma matrices $\Gamma^N = V_A^N \Gamma^A$, so that the fermion action is,

$$\mathcal{S}_\Psi = \int d^4x \int dy \sqrt{G}\{i\overline{\Psi}\Gamma^N D_N \Psi - g_j\Phi_j\overline{\Psi}\Psi\}. \tag{11}$$

The resulting Dirac equation is,

$$\left[\gamma^5\partial_y + ie^\sigma\gamma^\mu\partial_\mu - 2\sigma'\gamma^5 - g_j\phi_j(y)\right]\Psi(x^\mu,y) = 0. \tag{12}$$

We again decompose Ψ into 4D chiral components, and obtain equations for the extra-dimensional profiles $f_{L,R}$. As long as the value $g_j\Phi_j^{\text{min}}$ is large enough, the conclusions are unchanged in the massless case (see eg. [9, 10]). For $m_n > 0$ we get the following equations,

$$-f_{L,R}^n{}'' + 5\sigma' f_{L,R}^n{}' + \left[2\sigma'' - 6\sigma'^2 + \tilde{W}_\mp\right] f_{L,R}^n = m_n^2 e^{2\sigma} f_{L,R}^n, \tag{13}$$

where $\tilde{W}_\pm = (g_j\phi_j)^2 \pm g_j\phi_j' \mp g_j\phi_j\sigma'$. The inclusion of gravity has meant that Eq. (13) is no longer simply a Schrödinger equation, so we can't directly apply our knowledge of 1D quantum mechanics. It is however possible to transform Eq. (13) into a Schrödinger equation. Specifically, we let $f_{L,R}^n = e^{2\sigma}\tilde{f}_{L,R}^n$, and change coordinates to $z(y)$ such that $\frac{dz}{dy} = e^\sigma$ (this is in fact a change to 'conformal coordinates', in which $ds^2 = e^{-2\sigma(y(z))}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2)$). Eq. (13) becomes,

$$\left[-\frac{d^2}{dz^2} + e^{-2\sigma}\tilde{W}_\mp\right]\tilde{f}_{L,R}^n = m_n^2\tilde{f}_{L,R}^n. \tag{14}$$

We thus identify the effective potential,

$$\tilde{W}_\pm^{\text{eff}} = e^{-2\sigma}\tilde{W}_\pm. \tag{15}$$

As $|y| \to \infty$, $\sigma \sim \mu|y|$, which in terms of z becomes $e^{-2\sigma} \sim 1/(\mu z)^2$ as $|z| \to \infty$. As $|z| \to \infty$, we have $\tilde{W}_\pm \to$ constant, and therefore the effective potential decays towards zero at large distances from the brane (see Fig. 1 for some specific cases). Indeed, it is an example of a 'volcano potential', familiar from analysis of the graviton sector [5]. Particles subjected to $\tilde{W}_\pm^{\text{eff}}$ are essentially free asymptotically, so there is a continuum of delta-function

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normalisable solutions for all $m_n^2 > 0$. We will now show that normalisability of $\tilde{f}_{L,R}^n$ implies appropriate normalisability of $f_{L,R}^n$, and conclude that our dimensionally reduced theory contains a continuum of fermions starting at zero mass.

The $\tilde{f}_L^n$ satisfy an ordinary Schrödinger equation with a continuum of eigenvalues, and therefore are delta-function orthonormalisable,

$$\int_{-\infty}^{\infty} dz \, \tilde{f}_L^{n*} \tilde{f}_L^{n'} = \delta(n - n'). \tag{16}$$

On the other hand, the normalisation condition for the $f_{L,R}^n$ can be derived by demanding that integrating the action in Eq. (11) over y leads to a properly normalised 4D kinetic term, viz.

$$\int d^4x \int dy \sqrt{G} \left\{ i\overline{\Psi} \Gamma^A V_A^\mu \partial_\mu \Psi \right\} = \int d^4x \int dn \, i \overline{\psi}^n \gamma^\mu \partial_\mu \psi^n. \tag{17}$$

Substituting in the expressions for the vielbein and metric, this condition becomes,

$$\int_{-\infty}^{\infty} dy \, e^{-3\sigma} f_L^{n*} f_L^{n'} = \delta(n - n'), \tag{18}$$

and similarly for the right-handed components (overlap integrals between right- and left-handed components do not enter). However, if we write f_L^n in terms of $\tilde{f}_L^n$ and use $dy = e^{-\sigma} dz$, we see that,

$$\int_{-\infty}^{\infty} dy \, e^{-3\sigma} f_L^{n*} f_L^{n'} = \int_{-\infty}^{\infty} dz \, \tilde{f}_L^{n*} \tilde{f}_L^{n'}. \tag{19}$$

So the normalisation integral for the f_L^n is equivalent to the normalisation integral for the $\tilde{f}_L^n$. Therefore there is a continuum of normalisable fermion modes in the theory, starting at zero 4D mass. Despite this, the zero modes still form an effective 4D theory at low energies. We can understand this as follows.

Flat space corresponds to $\sigma \equiv 0$, and we have seen that in this case the low-energy spectrum consists of a finite number of particles with discrete masses. The reason for this is that the effective potential of the analogue Schrödinger system, given in Eq. (7), asymptotes to a non-zero value; the discrete spectrum corresponds to modes bound in the potential well near $y = 0$.

Suppose now that σ is non-zero, but grows only very slowly with $|y|$. In this case, the effective potential in Eq. (14) approximates that of Eq. (7) near the brane, then decays towards zero as $|y| \to \infty$. We thus have a localised non-zero potential in the form of a narrow well flanked by wide barriers. The low-energy continuum eigenfunctions of such a

system will generically have very small amplitudes at the position of the well, due to the potential barrier which they must tunnel through.

So although arbitrarily light fermions will exist in the theory, their wavefunctions will be strongly suppressed at the position of the brane, where the zero modes reside. They are effectively 'localised at infinity'. This leads to a very small probability of these low-energy continuum modes interacting with the zero modes.

There is one more generic feature which we expect to occur. Certain discrete energies will resonate with the potential, and the corresponding states will thus have a much larger probability of being found on the brane. These are the remnants of the discrete bound states in the flat space case, and become coincident with them in the zero-gravity limit. What happens if one of these resonant modes is produced in a high-energy process on the brane? Any particle produced on the brane will have a wavefunction truly localised to the brane, and thus cannot correspond exactly to a single mode ψ^n, which has a wavefunction oscillatory as $z \to \infty$. Instead it will be a wavepacket made from the continuum modes, with a Fourier spectrum peaked around one of the resonances. Therefore it is not a true energy/mass eigenstate, and as the various components become out of phase, the wavefunction will leak off the brane. The particle then has some probability of escaping the brane, which justifies the moniker "quasi-stable" or "quasi-localised" for the resonant modes. It is these resonant quasi-localised states that are investigated in [12].

Quantitative calculations confirming the above conclusions will be given for a particular toy model in Section V.

IV. LOCALISED SCALAR FIELDS WITH GRAVITY

As well as fermions, it is desirable for model building purposes to be able to localise scalar fields to the wall. In flat space, the results are similar to those for fermions, except that the mass-squared of the lightest mode depends on parameters in the 5D theory [2] (whereas in the fermion case, it is always zero). It can even be arranged to be negative, so as to realise the Higgs mechanism in the low-energy theory. We will now examine the effects of gravity on these results.

We consider a scalar field Ξ described by the action,

$$\mathcal{S}_\Xi = \int d^4x \int dy \sqrt{G} \left\{ G^{MN} (\partial_M \Xi)^\dagger \partial_N \Xi - H(\Phi, \Xi) \right\}, \tag{20}$$

where H specifies the coupling of Ξ to itself and to the domain-wall. The linearised equation of motion for Ξ is given by,

$$\partial_M \left(\sqrt{G} \, G^{MN} \partial_N \Xi \right) + \sqrt{G} \, U(\phi_j) \Xi = 0, \tag{21}$$

where U is independent of Ξ, and defined by $\frac{\partial H}{\partial \Xi^\dagger} = U\Xi + \mathcal{O}(\Xi^2)$. We solve this exactly the same way as in the fermion case:

$$\text{Let} \quad \Xi(x^\mu, y) = \sum_n h^n(y) \xi^n(x^\mu), \tag{22}$$

where each ξ^n satisfies a 4D Klein-Gordon equation,

$$\partial^\mu \partial_\mu \xi^n + m_n^2 \xi^n = 0. \tag{23}$$

The analogue of Eq. (13) is then,

$$- h^{n\prime\prime} + 4\sigma' h^{n\prime} + U h^n = e^{2\sigma} m_n^2 h^n \tag{24}$$

We can convert this to a Schrödinger equation by again going to the conformal coordinate z, as well as making the substitution $h^n = e^{\frac{3}{2}\sigma} \tilde{h}^n$. This yields,

$$- \frac{d^2 \tilde{h}^n}{dz^2} + \left[-\frac{3}{2} \frac{d^2\sigma}{dz^2} + \frac{9}{4} \left(\frac{d\sigma}{dz} \right)^2 + e^{-2\sigma} U \right] \tilde{h}^n = m_n^2 \tilde{h}^n. \tag{25}$$

As $|z| \to \infty$, $\sigma \sim \log|z|$, and $U \to constant$, so we can see immediately that, as in the fermion case, the effective potential decays towards zero far from the brane. Therefore the low-energy scalar spectrum also contains a continuum of modes of arbitrarily small mass, which are properly normalisable, as can be shown by a calculation analogous to that described above for the fermions.

If $U \equiv 0$, the above equation is in fact identical to that satisfied by 4D gravitons in the background of Eq. (9) (see eg. Ref [15]). In this case then, we know that there is a single zero mode, followed by a continuum of modes starting arbitrarily close to $m_n^2 = 0$. The low lying continuum modes are strongly suppressed on the brane; for example, their contribution to a static potential generated by Ξ exchange between two sources on the brane separated by r, is suppressed by $1/(\mu r)^2$ relative to the contribution of the zero mode.

For non-zero U, the spectrum is modified from the graviton case, the significant difference being the possible introduction of resonant modes (in the absence of fine-tuning of parameters, there will no longer be a zero mode). As in the fermion case, these resonant modes correspond to the discrete bound modes in the corresponding gravity-free theory and we expect the first of these modes to occur for $m_n \sim \mu$. Unlike the fermion case, if appropriate coupling to the domain-wall is included, such that U makes some negative contribution to the effective potential, then there may be bound state solutions with $m_n^2 < 0$, as in the gravity-free case [2]. This signals an instability in the system, and implies that Ξ is non-zero in the stable background configuration. In this case we would have to instead solve the coupled Einstein and Klein-Gordon equations including the Φ_j fields *and* Ξ. This setup can be used for interesting model building, in which a symmetry is broken on the brane but restored in the bulk. This idea has been used in the flat space case in Ref. [16]. We sketch the reasoning following Ref. [17]. Take the scalar potential

$$H(\Phi, \Xi) = (g'\Phi^2 - u^2)\Xi^\dagger\Xi + \tau(\Xi^\dagger\Xi)^2 \tag{26}$$

where we have specialised to a single background field Φ, and we assume $g'\Phi^{\mathrm{min}} - u^2 > 0$ such that $(\Phi, \Xi) = (\pm\Phi^{\mathrm{min}}, 0)$ are still the global minima of the potential, and we must have $\Xi \to 0$ as $|y| \to \infty$. If Φ forms a domain wall, then $\Phi \sim 0$ inside the wall, so that the leading term of $H(\Phi, \Xi)$ is $\sim -u^2\Xi^\dagger\Xi$, suggesting that the $\Xi = 0$ solution is unstable there. This will show up as a negative eigenvalue $m_n^2 < 0$ in equation (25), and solving for a consistent set of background solutions will yield a background Ξ that is peaked on the brane and tending to zero in the bulk. Putting Ξ in a non-trivial representation of some gauge group will induce spontaneous breaking of that group on the brane, a mechanism which can be used, for example, to realise the standard model Higgs mechanism on the brane.

In the stable case then, asymptotically U will approach some constant positive value U_0. As $|z| \to \infty$, we can approximate the effective potential V_{eff} as follows:

$$V_{\mathrm{eff}} \sim \frac{1}{z^2}\left(\frac{15}{4} + \frac{U_0}{\mu^2}\right). \tag{27}$$

Again we can appeal to the results of Ref. [15], where it is shown that for a potential that behaves asymptotically as $\alpha(\alpha + 1)/z^2$, the amplitudes of modes with small m_n are suppressed by $(m_n/\mu)^{\alpha-1}$. Therefore the coupling to the domain-wall actually reduces the effect of the continuum modes on low energy physics.

V. TOY MODEL CALCULATION

The above conclusions can be illustrated concretely by finding numerical results for a specific case. We will make use of a background solution found in Ref. [18] in which a single real scalar field forms the domain-wall. The stability of this solution is demonstrated in Ref. [18]. The solutions for the warp factor and scalar field are[2],

$$\sigma(y) = a \log(\cosh(ky))$$
$$\phi(y) = D \arctan(\sinh(ky)), \tag{28}$$

where $a = D^2/12M^3$ is proportional to the 5D Newton's constant, and the solution corresponds to a 5D cosmological constant given by $\Lambda = -D^4k^2/6M^3$. The scalar field potential which admits this solution is,

$$V(\Phi) = 6ak^2M^3(1+4a)\sin^2\left(\frac{\phi}{D} - \frac{\pi}{2}\right). \tag{29}$$

We'd like to study a fermion field Ψ in the above background to illustrate the existence and suppression of the low-lying continuum modes. We again impose a global $U(1)$ symmetry $\Psi \to e^{i\theta}\Psi$ so that Ψ only couples to the background via a term $g\Phi\overline{\Psi}\Psi$. For the sake of examining interactions later, we also include a scalar field Ξ, which $U(1)$ acts on via $\Xi \to e^{2i\theta}$, to mediate interactions between Ψ quanta[3], and take the action describing these two fields to be

$$\mathcal{S}_{\Psi\Xi} = \int d^4x \int dy \sqrt{G}\Big[i\overline{\Psi}\Gamma^M\partial_M\Psi - g\Phi\overline{\Psi}\Psi + \partial^M\Xi^\dagger\partial_M\Xi - g'\Phi^2\Xi^\dagger\Xi - u^2\,\Xi^\dagger\Xi - \tau(\Xi^\dagger\Xi)^2$$
$$- \lambda(\Xi\overline{\Psi}\Psi^c + \text{h.c.})\Big], \tag{30}$$

where $\Psi^c = \Gamma^2\Gamma^5\Psi^*$. Note that the full action includes the background given by Eq. (8).

For our background solution to remain stable, $\Xi = 0$ must be the stable solution i.e. Eq. (25) must not have any negative eigenvalues. Choosing $u^2 > 0$ suffices to guarantee this.

The effective Schrödinger equation for the fermion field is found as described in Section III. For various values of a, the resulting effective potential felt by the left chiral component of

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[2] The functional form of this solution appears different to that given in Ref. [18], but it is in fact equivalent.

[3] It is necessary to introduce a field other than Φ, because the fermion zero mode is chiral, and thus does not interact with Φ via the term $g\Phi\overline{\Psi}\Psi$. Additionally, while the modes of Φ mix with scalar gravitational degrees of freedom, the global $U(1)$ symmetry prevents such a mixing of Ξ modes.
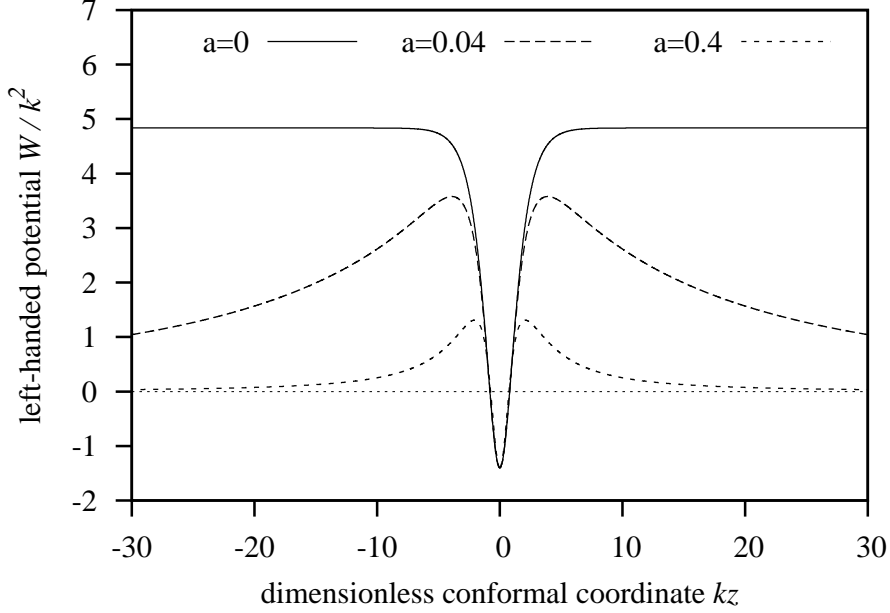
FIG. 1: An example of the effective Schrödinger potential, $\tilde{W}_-^{\text{eff}}$, which traps a left-handed fermion field. The three graphs correspond to no gravity ($a = 0$), 'weak' gravity ($a = 0.04$), and 'strong' gravity ($a = 0.4$). The horizontal line is $W = 0$. All plots have $gD = 1.4\,k$.

the fermion is plotted in Fig. 1. It does indeed asymptote to zero when gravity is included, implying the existence of a continuum of arbitrarily light modes. There is of course a zero mode which is localised to the brane – all other modes are however oscillatory at infinity.

We wish to quantify our argument that the light continuum modes do not overly influence physics on the brane. The dominant process by which they could be detected in our toy model is two zero mode particles annihilating to produce two continuum particles via exchange of a Ξ quantum, so this is the process we will consider. As explained at the end of Section III, a Ξ quantum produced on the brane will not correspond to a single mass mode, but will be a wavepacket initially localised on the brane. The creation of such a wavepacket on the brane and the ensuing shape of the wavepacket will be a complicated issue, and not considered here. Instead we will simply take sech(kz) as a typical localised profile[4] and assume that a Ξ quantum is produced with the extra-dimensional wavefunction $\tilde{h}(z) = \sqrt{k/2}\,\text{sech}(kz)$. We have computed the Fourier decomposition of $\tilde{h}(z)$ in terms of the mass eigenmodes (the eigenfunctions of Eq. (25)); the spectrum is sharply peaked at a

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[4] Results should be almost identical for any profile which decays exponentially beyond $|z| \sim 1/k$.

mass corresponding to the first resonant mode, as expected. We now proceed to calculate the effective coupling of the fermion modes to this particle in the dimensionally reduced theory. This will give us a quantitative estimate of the likelihood of continuum fermion modes being produced by on-brane dynamics through s-channel annihilation. It will also be a valid estimate for t-channel scattering of localised zero modes with bulk continuum modes.

The effective coupling constant between the fermion modes of Ψ and the localised Ξ particle is given by the 5D Yukawa coupling constant multiplied by the overlap integral of their extra-dimensional wavefunctions. For the fermion mode with extra-dimensional dependence $f^n(y)$, the coupling will be

$$
\begin{aligned}
\lambda_n^{(4)} &= \lambda \int dy\, e^{-4\sigma}\, h(y) \left(f^n(y)\right)^2 \\
&= \lambda \int dz\, e^{\frac{1}{2}\sigma} \tilde{h}(z) \left(\tilde{f}^n(z)\right)^2 \\
&= \lambda \int dz\, e^{\frac{1}{2}\sigma} \frac{\left(\tilde{f}^n(z)\right)^2}{\cosh ky}.
\end{aligned}
\tag{31}
$$

The results for the case $a = 0.04$ are plotted in Fig 2, contrasted with the results in the gravity-free case[5]. It is clear that the 4D coupling constants go quickly to zero for modes with masses much less than the inverse width k of the domain-wall.

Such behaviour is of course easy to understand, based on the discussion of Section III. Modes with energy much less than k see a wide potential barrier preventing them from penetrating to the brane, where the Ξ particle resides. At an energy approximately equal to k, we see the first resonant mode, which does not suffer the generic suppression near the brane. Continuum modes with energy above the barrier height ($\sim 5k^2$ for the gravity-free case and $\sim 3.5k^2$ for weak gravity) are free to roam in the vicinity of the brane, hence their coupling is of order unity. We have explicitly plotted the profiles of a resonant mode and a (slightly) off-resonant mode in Fig. 3, to illustrate the amplification of one and suppression of the other on the brane.

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[5] Note that what is plotted is really "interaction strength per continuum mode" with mode energy used on the horizontal axis to label a particular mode number. An integral over some finite range of modes is required to yield a finite on-brane effect.
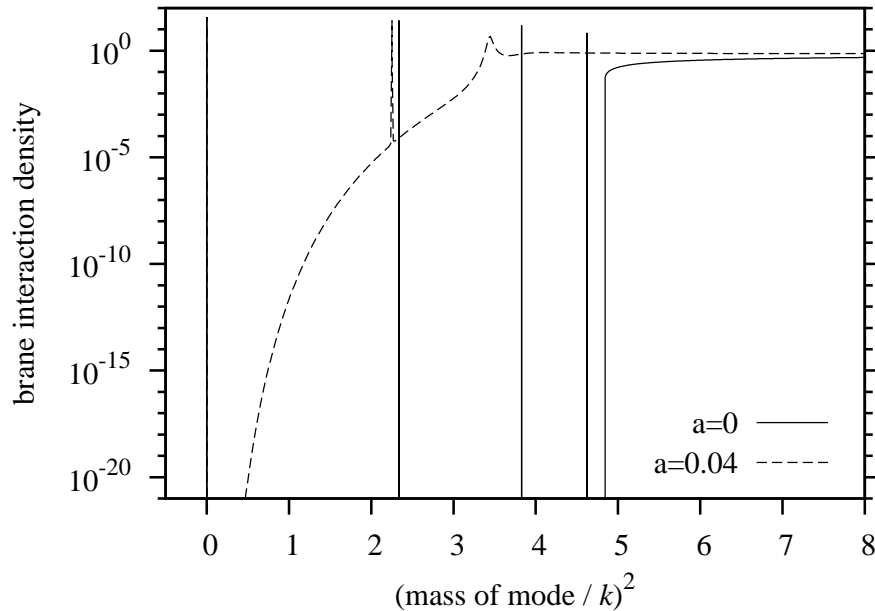
FIG. 2: The "interaction per continuum mode", $\lambda_n^{(4)}/\lambda\sqrt{k}$, for continuum modes interacting with a typical bound mode on the brane. Both the gravity-free ($a = 0$) and 'weak' gravity ($a = 0.04$) cases are shown. The fermion zero mode remains bound in the presence of gravity (hidden by the gravity-free plot), whilst a continuum is introduced for all positive energies. It is clear that at energies well below k, the continuum modes are essentially decoupled from those on the brane. The coupling becomes relatively strong for energies greater than the maximum of the effective potential. The fermion coupling strength is $gD = 1.4\,k$.

VI. CONCLUSION

We have shown that when a fermion field is coupled to a gravitating scalar domain-wall in 5D, the spectrum of the low-energy effective 4D theory consists of a massless fermion of fixed chirality, and a continuum of states with all possible masses $m > 0$. The massless mode is bound to the brane, while the continuum modes are oscillatory far from the brane. A scalar field coupled to the same background also yields a continuum of arbitrarily light states.

Coupling constants in the low energy theory will be determined by overlap integrals between the extra-dimensional profiles of the fields involved. Generically, continuum modes are strongly suppressed on the brane, and thus should interact only very weakly with the zero modes and other localised fields. We have demonstrated this effect by explicitly computing
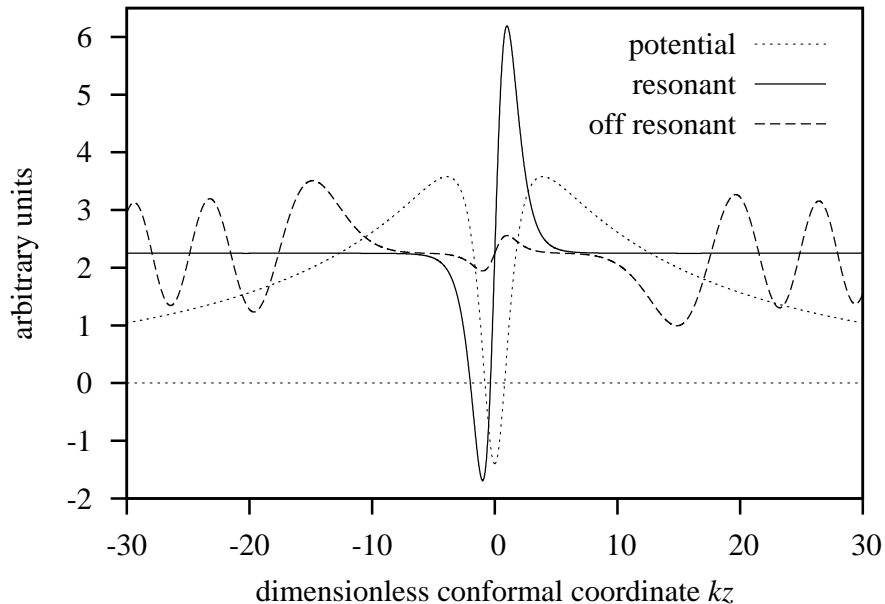
14

FIG. 3: The extra-dimensional profiles of the resonant mode at $(E/k)^2 \simeq 2.3$, and a mode off-resonance by 2.0×10^{-4} in units of $(E/k)^2$. We are in the 'weak' gravity case with $a = 0.04$. The profiles are not plotted on the same scale; in reality, each is normalised to the same amplitude *at infinity* (since the normalisation condition is dominated by the behaviour of the wavefunction at infinity). Thus the contrast is much more dramatic even than it appears here.

the overlap integrals for a typical toy model. It should be possible within specific models to arrange for the integrated effects of these modes to be small enough not to contradict known low-energy phenomenology. Nevertheless, at higher energies it may be important to consider the effects of such modes.

There will be a finite number of resonant modes which *will* manifest on the brane; these are the remnants of the bound states of the analogue Schrödinger system in the non-gravitating case. The lowest of these modes will have a mass approximately equal to the inverse width of the domain-wall, which would need to be sufficiently large in a realistic model.

Note that we have assumed throughout that the 4D metric is Minkowskian. Cosmologically it may be desirable to allow it to be de Sitter; in this case by dimensional analysis the continuum matter modes will begin at a mass $\sim \sqrt{\Lambda_4}$, where Λ_4 is the effective cosmological constant on the brane (this effect is demonstrated explicitly for gravitons in [19]). For our

universe this is sufficiently small to be negligible for collider phenomenology.

The analysis presented in this paper would form the basis of an investigation of the low-energy phenomenology of a realistic model; for example of that presented in [13].

VII.   ACKNOWLEDGEMENTS

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